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Graphs and free partially commutative monoids

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Abstract

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The relations between graphs and their corresponding free partially commutative monoids are studied. The connecting link is the so-called clique-polynomial which turns out to be the Möbius-function of the free partially commutative monoid. Various properties of this polynomial are studied. In particular the case when it is irreducible is characterized.

1. Introduction

Let A be a finite set and $\binom{A}{2}$ the set of all two-element subsets of A. Every subset θ of $\binom{A}{2}$ defines a free partially commutative monoid by allowing two letters to commute if and only if $\{a, b\} \in \theta$. More precisely, the free partially commutative monoid over A with commutation relation θ is defined to be

$$M(A, \theta) = A^*/\{ab = ba \mid \{a, b\} \in \theta\}, \tag{1}$$

where A^* is the free monoid over the set A with unit-element 1. Free partially commutative monoids have been introduced in [2] to represent certain combinatorial objects like flows and rearrangements. During the last ten years free partially commutative monoids also occurred in theoretical computer science as a model of concurrency and were studied by various authors [3, 6]. Since then elements of $M(A, \theta)$ usually are called traces, subsets of $M(A, \theta)$ are called trace-languages.

There is a lot of combinatorics involved in free partially commutative monoids. There are different ways of presenting $M(A, \theta)$ which give rise to different ways of representing the elements of $M(A, \theta)$. In Section 2 we give a new and unified approach to some well-known representation theorems for $M(A, \theta)$. We shall use two binary operations $\oplus$ and $\otimes$ for graphs and two functors from the category of partially ordered graphs to the category of free partially commutative monoids. The operations $\oplus$ and $\otimes$ seem to be very natural, because they translate into very simple counterparts in the category of free partially commutative monoids. The set $\mathcal{G}$ of all finite graphs turns out to have an interesting algebraic structure with these operations.

A very powerful tool for the study of the combinatorial properties of $M(A, \theta)$ is its Möbius-function. It was shown in [2] that the Möbius-function μ_M of $M(A, \theta)$ is the following polynomial in $\mathbb{Z}\langle\langle M \rangle\rangle$.

Consider θ as the edge set of an undirected graph G with vertex set A. A subset C of A is called a *clique* of the graph $G=(A, \theta)$ if $\binom{C}{2} \subseteq \theta$. Define the clique-polynomial $p(G)$ of G to be

$$p(G)=\sum (-1)^n a_1 a_2 \ldots a_n, \tag{2}$$

where the sum is taken over all cliques $\{a_1, a_2, \ldots, a_n\}$ of G. Then the Möbius-function of M is $\mu_M = p(G)$.

In Section 3 we study some of the properties of $p(G)$. The operations $\oplus$ and $\otimes$ have again very simple interpretations in polynomial operations. In particular we shall see that $p(G)$ is irreducible iff the complementary graph of G is connected.

In Section 4 we construct different systems of representatives for $M(A, \theta)$. We generalize the well-known concepts of Foata's normal form and alphabetic normal form by defining them relative to some not necessarily total order. Moreover, we derive a so-called Left-Right normal form, which is closely related to some order on the graph, implicitly defined by some representation of $M(A, \theta)$. This normal form can be calculated without making explicit use of some order but takes into account the representation of $M(A, \theta)$ only.

In [5], Diekert characterized those free partially commutative monoids with a particularly simple Möbius-function in terms of some order-property of the defining graph. In Section 5 we study this property in more detail.

Our considerations will show that to know the structural properties of all these concepts it is sufficient to know them on a generating set for the algebra $\mathcal{G}$ of all finite graphs. A minimal generating set for this algebra is exhibited.

2. Presentations of $M(A, \theta)$

For graphs $G=(A, \theta)$ and $H=(B, \phi)$ we say that H is a *subgraph* of G, if $B \subseteq A$, and $\phi \subseteq \theta$. For every graph $G=(A, \theta)$ we have the (edge-) *complementary* graph $\bar{G}=(A, \bar{\theta})$, where $\bar{\theta}=\binom{A}{2}-\theta$. The corresponding monoid $M(\bar{G})=M(A, \bar{\theta})$ is also denoted by $\bar{M}(A, \theta)$. The reason for this will be clear from Lemma 2.3(i). For example, $M(A, \emptyset)=\bar{M}(A, \binom{A}{2})=A^*$, and $M(A, \binom{A}{2})=\bar{M}(A, \emptyset)=A^{\oplus}$, the free commutative monoid over A.

Let $G=(A, \theta)$ be a graph and B a subset of A. The graph

$$H=\left(B, \theta \cap \binom{B}{2}\right) \tag{3}$$

is called an *induced* subgraph and frequently denoted by (B, θ). We also say that G is an *extension* of H. It is clear that H is an induced subgraph of G iff H is a subgraph of G and $\bar{H}$ is a subgraph of $\bar{G}$. It can be easily seen that in this case $M(B, \theta)$ is a submonoid and a homomorphic image of $M(A, \theta)$ as well.

Furthermore, we define for graphs $G=(A, \theta)$ and $H=(B, \phi)$ their *direct sum* to be

$$G \oplus H=(A+B, \theta+\phi), \tag{4}$$

where "+" for sets means disjoint union, and the "*product*"

$$G \otimes H=\overline{\bar{G} \oplus \bar{H}}. \tag{5}$$

Then it is easily checked that $G \oplus H=\overline{\bar{G} \otimes \bar{H}}$ and $G \otimes H=\{A+B, \psi\}$ with edge set $\psi=\theta+\phi+\{\{a, b\} \mid a \in A, b \in B\}$, and we obtain the following identities:

$$M(G \oplus H)=M(G) * M(H) \quad \text{(free product)}, \tag{6}$$

$$M(G \otimes H)=M(G) \times M(H) \quad \text{(direct product)}. \tag{7}$$

Both these operations are commutative and associative, and the graph $\Phi=(\emptyset, \emptyset)$ is the neutral element for $\oplus$ and $\otimes$. As an immediate consequence of these equalities we can derive the following.

Lemma 2.1. (1) *Let G be a graph and $G=Z_1 \oplus Z_2 \oplus \cdots \oplus Z_n$ the decomposition of G into its connected components, $\bar{G}=Y_1 \oplus Y_2 \oplus \cdots \oplus Y_m$ the decomposition of $\bar{G}$ into its connected components. Then $M(G) \cong M(Z_1) * \cdots * M(Z_n)$ and $M(G) \cong M(\bar{Y}_1) \times \cdots \times M(\overline{Y_m})$.*

(2) *If $M(G) \cong M_1 \times M_2$, then there are unique subgraphs H and K of G such that $G=H \otimes K$ and $M_1 \cong M(H)$, $M_2 \cong M(K)$.*

(3) *If $M(G) \cong M_1 * M_2$, then there are unique subgraphs H and K of G such that $G=H \oplus K$ and $M_1 \cong M(H)$, $M_2 \cong M(K)$.*

Proof. (1) $\bar{G}=Y_1 \oplus Y_2 \oplus \cdots \oplus Y_n \Leftrightarrow G=\overline{Y_1} \otimes \overline{Y_2} \otimes \cdots \otimes \overline{Y_n}$.

(2) Let $G=(A, \theta)$, $M(G) \cong M_1 \times M_2$, and $R=\{(l_a, r_a) \mid a \in A\}$ the subset of $M_1 \times M_2$ corresponding to A. R is a minimal generating set for $M_1 \times M_2$, hence the sets $A_1=\{a \in A \mid l_a \neq 1\}$ and $A_2=\{a \in A \mid r_a \neq 1\}$ are disjoint and $A=A_1+A_2$. For arbitrary letters $a \in A_1$ and $b \in A_2$ we find $(l_a, 1) \cdot (1, r_a)=(1, r_a) \cdot (l_a, 1)$, hence $\{a, b\} \in \theta$, whereas for $a, b \in A_1$ we have $(a, b) \in \theta$ iff $(l_a, 1) \cdot (l_b, 1)=(l_b, 1) \cdot (l_a, 1)$, and similarly for $a, b \in A_2$. We denote $H=(A_1, \theta)$ and $K=(A_2, \theta)$. Then $G=H \otimes K$ and $M_1 \cong M(H)$, $M_2 \cong M(K)$.

(3) Let $G=(A,\theta)$ be given, $M(G)\cong M_1 * M_2$, and, similarly as in (2), $R=\{l_{a,1}r_{a,1}l_{a,2}r_{a,2}\cdots l_{a,n(a)}r_{a,n(a)} \mid a\in A\}$ the subset of $M_1 * M_2$ corresponding to A. Put $A_1=\{a\in A \mid \exists j\ l_{a,j}\neq 1\}$ and $A_2=\{a\in A \mid \exists j\ r_{a,j}\neq 1\}$. By an argument as in (2) we obtain $A=A_1+A_2$ and for $a\in A_1$ and $b\in A_2$ $\{a,b\}\notin\theta$. This implies for $H=(A_1,\theta)$ and $K=(A_2,\theta)$: $G=H\oplus K$ and $M_1\cong M(H)$, $M_2\cong M(K)$. $\square$

To give an example, let $G=C_A$, the clique over the set $A=\{a_1,a_2,\ldots,a_n\}$. For every vertex $a\in A$ we denote the graph $C_{\{a\}}=(\{a\},\emptyset)$ by a and $M(a)$ by a^*. By the lemma we obtain from the sum $\bar{G}=(A,\emptyset)=a_1\oplus a_2\oplus\cdots\oplus a_n$ the product representation $A^{\oplus}=a_1^*\times a_2^*\times\cdots\times a_n^*$.

It is immediate that a graph G can be written $G=H\oplus K$ with proper induced subgraphs H and K iff G is not connected, and $G=H\otimes K$ iff $\bar{G}$ is not connected. We call H and K *parts* of G, and parts of parts of G are again parts of G. A graph is called *indecomposable*, if it is connected and its complement is connected, i.e. if it has no proper parts. In particular, $\Phi=(\emptyset,\emptyset)$ is an indecomposable graph.

We define $\oplus,\otimes$-*expressions* inductively by the following conditions:

- every indecomposable graph is a $\oplus,\otimes$-expression
- if G and H are $\oplus,\otimes$-expressions, then $G\oplus H$ and $G\otimes H$ are $\oplus,\otimes$-expressions.

We say that a graph G has a *primitive* $\oplus,\otimes$-expression, if G can be described by a finite $\oplus,\otimes$-expression in which all indecomposable graphs are one-element graphs. It is clear that a $\oplus,\otimes$-expression for G can be converted into a $\oplus,\otimes$-expression for $\bar{G}$ by replacing every occurrence of $\oplus$ by $\otimes$ and vice versa and every indecomposable part occurring in the $\oplus,\otimes$-expression by its complement.

Besides $\oplus$ and $\otimes$, the following operations for graphs are convenient:

$$G\cup H=(A\cup B,\theta\cup\phi),\qquad G\cap H=(A\cap B,\theta\cap\phi).$$

We note that $\oplus$ and $\otimes$ do not distribute in one or the other order. Moreover, for all non-empty graphs G, H, and K, we have $(G\oplus H)\otimes K\neq(G\otimes K)\oplus(H\otimes K)$, because every graph has a unique *normal form*, as is shown by the next lemma; but $(G\oplus H)\otimes K=(G\otimes K)\cup(H\otimes K)$ for arbitrary graphs G, H, and K.

Lemma 2.2. *Up to commutativity and associativity of $\oplus$ and $\otimes$, every graph has a uniquely determined $\oplus$, $\otimes$-expression, the normal form of the graph.*

Proof. If G is not connected, then $\bar{G}$ is connected, hence the root of the tree representing the $\oplus,\otimes$-expression is uniquely determined and the corresponding parts are uniquely determined up to commutativity and associativity of $\oplus$ and $\otimes$. Applying this argument recursively to the corresponding subtrees gives the desired result. $\square$

The normal form of G can be represented by a binary tree, representing the $\oplus,\otimes$-expression. By the associativity of $\oplus$ and $\otimes$, this tree can be converted into a (not necessarily binary) alternating tree, i.e. every non-leaf successor of a $\oplus$-node is a $\otimes$-node and vice versa. We can make this alternating normal form of G unique if we put a suitable partial order onto the set of all parts of G. This can be done, for example, in the following way. Put a total order on A. If $H=(B,\phi)$ and $K=(D,\psi)$ are parts, then $H<K$ iff $\min(B)<\min(D)$ and $H\cap K=\Phi$. This defines a partial order on the set of all parts of G. Ordering the subtrees of every node in the alternating $\oplus,\otimes$-tree for G according to this order, we obtain the *alternating normal form* of G. For example, if the $\oplus,\otimes$-tree of the alternating normal form of G has height at most two, then G is an n-partite graph or a disjoint union of cliques.

We remark that it is decidable whether a graph G has a primitive $\oplus,\otimes$-expression. The smallest example for a graph which does not possess a primitive $\oplus,\otimes$-expression is the graph L_4, shown in Fig. 1, since it is connected and its complement is again connected.

L_4:

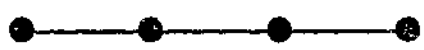


Fig. 1.

Thus we can consider the set of all finite graphs as the free algebra of type $(2,2,0)$ with two associative and commutative operations and a common neutral element, generated by the set of indecomposable graphs over finite subsets of a countable set. On this algebra we have an involution $G\mapsto\bar{G}$ which is induced by an involution on the indecomposable graphs, and the laws $G\oplus H=\overline{\bar{G}\otimes\bar{H}}$ and $G\otimes H=\overline{\bar{G}\oplus\bar{H}}$. The one-point graphs and Φ are fixed points for this involution. For two graphs $G=(A,\theta)$ and $H=(A,\phi)$ we have $\overline{G\cap H}=\bar{G}\cap\bar{H}$ if G and H are cliques or $\bar{G}$ and $\bar{H}$ are cliques.

There is a second normal form for every graph $G=(A,\theta)$, which we call the *canonical form* of G. Up to commutativity and associativity of $\cup$ and $\otimes$ and idempotency of $\cup$ every graph is uniquely determined by its maximal cliques, i.e. expressible as union of graphs of the form $a_1\otimes a_2\otimes\cdots\otimes a_m$ for some $m\in\mathbb{N}$, and $a_i\in A$ for $1\leqslant i\leqslant m$. If $C_1,\ldots,C_n$ are the maximal cliques of G, then G can be written $G=(C_1,\binom{C_1}{2})\cup\cdots\cup(C_n,\binom{C_n}{2})$. To simplify notation we write the canonical form of $G=C_1\cup\cdots\cup C_n$.

Let $G=(A,\theta)$ and $H=(B,\phi)$ be graphs. A *graph-morphism* $f:G\to H$ is a mapping $f:A\to B$ with the property

$$\forall a,b\in A\quad (\{a,b\}\in\theta\wedge f(a)\neq f(b)\Rightarrow\{f(a),f(b)\}\in\phi). \tag{8}$$

f is an injective graph-morphism if $f:A\to B$ is injective, and f is a surjective graph-morphism if the mapping $f:A\to B$ is surjective and the induced mapping $\binom{f}{2}:\binom{G}{2}\to 2^B$ has the property $\phi\subseteq\binom{f}{2}(\theta)$. As usual, isomorphisms are injective and surjective morphisms. It should be noted that for every injective graph-morphism $f:G\to H$, the inverse mapping $f^{-1}:\overline{f(G)}\to\bar{G}$ is again an injective graph-morphism, where $f(G)=(f(A),\phi)$.

Since it is a mapping between the sets of vertices, every graph-morphism $f: G \to H$ can be considered as a labelling function, and thus the pair (G, f) represents a labelling of the vertices of G. Vice versa, every labelling of a graph $G = (A, \theta)$ is a mapping $f: A \to B$ into some set B of labels which can be assumed to be surjective without loss of generality and thus defines a graph-morphism f from G onto the graph (B, ϕ) with $\phi = \{\{f(a), f(b)\} \mid a, b \in A, f(a) \neq f(b)\}$. The interest in graph-morphisms comes from the close relationship between graph-morphisms and monoid-morphisms of the corresponding free partially commutative monoids which is summarized by the following lemma.

Lemma 2.3. (1) *Every graph-morphism $f: G \to H$ has a unique extension to a monoid-morphism $M(f): M(G) \to M(H)$, and every total order on the vertices of G defines a monoid-morphism $\bar{M}(f): \bar{M}(H) \to \bar{M}(G)$.*

(2) *If f is injective, then every $\bar{M}(f)$ is surjective.*

(3) *If f is surjective, then $M(f)$ is surjective and every $\bar{M}(f)$ is injective.*

Proof. Let $G = (A, \theta)$ and $H = (B, \phi)$. Then $f: A \to B$ extends uniquely to a monoid-morphism $f: A^* \to B^*$ and by (8), f induces a monoid-morphism $M(f): M(G) \to M(H)$.

Define a morphism $f': B^* \to A^*$ by $f'(b) = \prod \{a \mid f(a) = b\}$, where for every $b \in B$ the order of the letters in the product is according to the given total order on the vertices of G, and the empty product is 1 for convenience. f is a graph-morphism, hence $\{b, b'\} \notin \phi$ implies $\{a, a'\} \notin \theta$ for all a and a' such that $f(a) = b$ and $f(a') = b'$, and we have in $\bar{M}(G)$:

$$\{b, b'\} \notin \phi, b \neq b' \Rightarrow f'(bb') = f'(b)f'(b') = f'(b')f'(b) = f'(b'b). \tag{9}$$

Thus f' induces a morphism $\bar{M}(f): \bar{M}(H) \to \bar{M}(G)$ and we have (1). If f has the additional property that $f^{-1}(b)$ is a clique in G for every $b \in B$, then $\bar{M}(f)$ is independent of the order chosen in $f'(b)$ and every total order on G defines the same morphism $\bar{M}(f)$.

If $f: G \to H$ is injective, then $\binom{f}{2}: \binom{A}{2} \to \binom{B}{2}$ and $f: A^* \to B^*$ are injective, and $f(A^*) = B^*/\{b = 1 \mid b \notin f(A)\}$. Hence, $\bar{M}(f)$ is surjective, and we have (2).

If $f: G \to H$ is surjective, then $f: A^* \to B^*$ is surjective and B^* is isomorphic to $A^*/\{a = b \mid a, b \in A, f(a) = f(b)\}$. Hence we can see that $M(f): M(G) \to M(H)$ is a surjective morphism, in fact it is an amalgamation of letters. To see that $\bar{M}(f)$ is injective, we note the following fact which is a consequence of (9) and the surjectivity of f:

(*) For every pair a, b of different letters of B we have $ab = ba$ in $\bar{M}(H)$ if and only if every letter of $f^{-1}(a)$ commutes in $\bar{M}(G)$ with every letter of $f^{-1}(b)$.

Now let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ be elements of B such that

$$\bar{M}(f)(x_1 \cdot x_2 \cdot \ldots \cdot x_n) = \bar{M}(f)(y_1 \cdot y_2 \cdot \ldots \cdot y_m).$$

This means: $f'(x_1)\cdot f'(x_2)\cdot\ldots\cdot f'(x_n)=f'(y_1)\cdot f'(y_2)\cdot\ldots\cdot f'(y_m)$ in $\bar{M}(G)$. Every letter occurring on the left side with some multiplicity must also occur on the right side with the same multiplicity and vice versa. Hence from the equivalence

$$f^{-1}(x_i)\cap f^{-1}(y_i)\neq\emptyset \iff x_i=y_j,$$

we obtain $n=m$ and $(x_1,x_2,\ldots,x_n)$ must be a permutation of $(y_1,y_2,\ldots,y_n)$. From $(*)$ it follows immediately that $(x_1\cdot x_2\cdot\ldots\cdot x_n)=(y_1\cdot y_2\cdot\ldots\cdot y_n)$ in $\bar{M}(H)$ and (3) is shown. □

The following lemma gives a first insight into the structure of graph-morphisms.

Lemma 2.4. *Let $f: G\to H=Z_1\oplus Z_2\oplus\cdots\oplus Z_n$ be a graph-morphism, where $Z_1, Z_2, \ldots, Z_n$ are arbitrary subgraphs of H. Then $G=Y_1\oplus Y_2\oplus\cdots\oplus Y_n$, where Y_i denotes the subgraph of G induced on the subset $f^{-1}(Z_i)$, $i\in\{1,2,\ldots,n\}$.*

Proof. The lemma follows from the considerations (9) in the proof of Lemma 2.3(1). □

With these properties we can define the following derived graphs.

(1) For a given graph $G=(A,\theta)$ we define the *canonical equivalence* relation η on G by

$$(a,b)\in\eta \iff \{c\in A\mid(a,c)\in\theta\}\cup\{a\}=\{c\in A\mid(b,c)\in\theta\}\cup\{b\}.$$

For later use we denote the set $\{c\in A\mid(a,c)\in\theta\}$ by the symbol A_a. It can easily be checked that η is in fact an equivalence relation on the set A. Let $C_1, C_2,\ldots,C_n$ be the maximal cliques of G. We denote by $\Pi(C_i)$ the equivalence partitioning A into C_i and $A-C_i$. Then η can be written

$$\eta=\bigcap_{i\in\{1,\ldots,n\}}\Pi(C_i).$$

In fact, if $a,b\in A$ are such that a belongs to the maximal clique C_i and b does not, then $(a,b)\notin\eta$, and if $(a,b)\in\bigcap_{i\in\{1,\ldots,n\}}\Pi(C_i)$ and $\{a,c\}\in\theta$, then a and c belong to some common maximal clique C_i. This clique also contains b, hence $\{b,c\}\in\theta$ follows. Therefore every η-class U is a clique in G and for every pair U, V of distinct η-classes we find $U\oplus V$ or $U\otimes V$ as a subgraph of G. η can also be expressed as the largest equivalence relation (relative to set inclusion) on A such that every maximal clique is a union of η-classes.

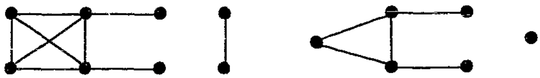
On the set of equivalence classes we define a graph structure θ/η by

$$\{[a]_\eta,[b]_\eta\}\in\theta/\eta \iff \exists a'\in[a]_\eta, \exists b'\in[b]_\eta \text{ such that } \{a',b'\}\in\theta.$$

This condition is equivalent to: for every $a'\in[a]_\eta$ and every $b'\in[b]_\eta$ we have

$\{a', b'\} \in \theta$. We call the graph $H = (A/\eta, \theta/\eta)$ the *skeleton* of G. For example, the skeleton of the graph in Fig. 2a is shown in Fig. 2b.

It is immediate that a graph is connected iff its skeleton is connected, and that the natural mapping is a surjective graph-morphism from G onto its skeleton H and from $\bar{G}$ onto $\bar{H}$. By part (3) of Lemma 2.3 we deduce that $M(H)$ is a homomorphic image of $M(G)$ and $M(H)$ is a submonoid of $M(G)$. If for some graph G the equivalence relation η is reduced to the diagonal on A, then G is called *skeletal*.



(a) (b)

Fig. 2.

(2) Let again $G = (A, \theta)$ be a given graph and B a finite set, such that a mapping $f: A \to 2^B$ exists with the property

$$\forall a, b \in A \quad (a \neq b \Rightarrow f(a) \cap f(b) = \emptyset).$$

We define a graph structure H on B by taking as maximal cliques of H the set-unions $\bigcup_{a \in C} f(a)$, where C is a maximal clique of G. Then G and H have the same skeleton. In particular, if G is skeletal, then G is the skeleton of H.

The previous lemmas can be used to give simple proofs of the following well-known theorems, which are fundamental for the further studies of the combinatorial properties of free partially commutative monoids (see e.g. [3, 4, 6]).

Theorem 2.5. *If $C_1, C_2, \ldots, C_n$ are the maximal cliques of a given graph G, then $M(G)$ is a homomorphic image of $C_1^{\oplus} * C_2^{\oplus} * \cdots * C_n^{\oplus}$; to be precise, let $C_{i,j} = C_i \cap C_j$ for $1 \leq i < j \leq n$. Then $M(G)$ is the free product of the monoids $C_1^{\oplus}, C_2^{\oplus}, \ldots, C_n^{\oplus}$, amalgamated by the monoids $C_{i,j}^{\oplus}$.*

Proof. Let $H = C_1 \oplus C_2 \oplus \cdots \oplus C_n$ be the direct sum of the maximal cliques of G. The vertices of H can be identified with the pairs (a, i) where a is a vertex of G and $a \in C_i$. The mapping f which sends the vertex (a, i) of H to the vertex $a \in G$ is a surjective graph-morphism from H onto G. Hence, by Lemma 2.3(3), $M(f): M(H) \to M(G)$ is a surjective monoid-morphism, and the proof of Lemma 2.3(3) shows that $M(G)$ is isomorphic to the quotient $M(H)/\{((a, i) = (b, j)) \mid f(a, i) = f(b, j), i \neq j\}$. From Lemma 2.1 follows that

$$M(H) = M(C_1) * M(C_2) * \cdots * M(C_n) = C_1^{\oplus} * C_2^{\oplus} * \cdots * C_n^{\oplus}.$$

$\{((a, i) = (b, j)) \mid f(a, i) = f(b, j), i \neq j\} = \{((a, i) = (a, j)) \mid i \neq j\}$, hence factorizing this monoid by the set $\{((a, i) = (b, j)) \mid f(a, i) = f(b, j), i \neq j\}$ means identifying those letters which were split by the construction of H. For given i and j $(i \neq j)$ this set is characterized by $C_{i,j} = C_i \cap C_j$, which generates the submonoid $C_{i,j}^{\oplus}$ of $C_i^{\oplus}$ and of $C_j^{\oplus}$.

Thus the quotient $M(H)/\{((a,i)=(a,j))\mid i\neq j\}$ is isomorphic to the amalgamation of $C_1^{\oplus}*C_2^{\oplus}*\cdots*C_n^{\oplus}$ by the monoids $C_{i,j}^{\oplus}$ for $1\leq i<j\leq n$ and the theorem follows. $\square$

If we consider the complementary graph for G then the theorem reads as follows.

Theorem 2.6. *Let $G=(A,\theta)$ be a graph. If $C_1, C_2, \ldots, C_n$ are the maximal cliques of $\bar{G}$, then $M(G)$ is isomorphic to the submonoid of $C_1^*\times C_2^*\times\cdots\times C_n^*$ generated by the set of elements of the form $\iota(a)=(a_1,a_2,\ldots,a_n)$, where $a\in A$ and a_i is a if $a\in C_i$ and* 1 *otherwise.*

Proof. The proof goes along the same lines as the proof of the last theorem. Define $\bar{H}=C_1\oplus C_2\oplus\cdots\oplus C_n$; then $\bar{G}$ is a homomorphic image of $\bar{H}$ by the morphism h, which sends $(a,i)\in C_i$ to a. By Lemma 2.3(3), there is an injective monoid-morphism $\bar{M}(h):\bar{M}(\bar{G})\to\bar{M}(\bar{H})$, hence an injection $\bar{M}(h):M(G)\to M(H)$. Since $H=\overline{C_1}\otimes\overline{C_2}\otimes\cdots\otimes\overline{C_n}$, Lemma 2.1 gives the product representation

$$M(H)\cong C_1^*\times C_2^*\times\cdots\times C_n^*,$$

and the definition of $\bar{M}$ gives

$$\bar{M}(h)(a)=\prod\{(a,i)\mid a\in C_i\}=\iota(a). \qquad \square$$

Note that in fact $M(G)$ is a *subdirect* product of the $M(\overline{C_i})$ with $i=1,\ldots,n$.

For a given graph $G=(A,\theta)$ we call a subset L of A an *anticlique* of G if the graph $(L,\binom{L}{2})$ is a subgraph of $\bar{G}$, i.e. L is a clique of $\bar{G}$. This is equivalent to saying that $(L,\emptyset)$ is an induced subgraph of G. The theorem can then be stated as follows: $M(G)$ is a subdirect product of the free monoids generated by the vertex sets of the maximal anticliques of G.

From this theorem it follows immediately that $M(G)$ is *cancellable*, i.e. $uv=uw\Rightarrow v=w$ and $vu=wu\Rightarrow v=w$. Moreover, Theorem 2.6 is the basis for the solution of many problems, as for example solution of equations, finding systems of representatives, etc., by solving them componentwise and then composing the component solutions to a universal solution. For example, we can see that $M(G)$ is *locally finite* because submonoids and direct products of locally finite monoids are locally finite. (A monoid M is called locally finite if every $m\in M$ has only a finite number of factorizations $m=m_1\cdot\ldots\cdot m_k$ with $k\geq 0$ and $m_i\in M-\{1\}$ for all $i\in\{1,\ldots,k\}$. The empty product yields 1.) From Theorem 2.5 it is immediate that, for $G=(A,\theta)$ with canonical form $G=C_1\cup C_2\cup\cdots\cup C_n$, $M(G)$ is a *bi-unitary* submonoid of $C_1^{\oplus}*C_2^{\oplus}*\cdots*C_n^{\oplus}$, i.e. if $u,v,w\in C_1^{\oplus}*C_2^{\oplus}*\cdots*C_n^{\oplus}$ with $u\cdot v=w$, then $v,w\in M(G)$ implies $u\in M(G)$, and $u,w\in M(G)$ implies $v\in M(G)$.

The proofs of the theorems and of Lemma 2.1(2) and (3) can be slightly modified to obtain the following generalizations.

Theorem 2.7. *Let $G=(A,\theta)$ be a graph.*

(1) *If $G=G_1\cup G_2\cup\cdots\cup G_n$ is a representation of G as a union of some subgraphs of G, then $M(G)$ is isomorphic to the free product of the monoids $M(G_i)_{i=1,\ldots,n}$, amalgamated by the monoids $M(G_i\cap G_j)_{i,j=1,\ldots,n}$.*

(2) *If $\bar{G}=G_1\cup G_2\cup\cdots\cup G_n$ is a representation of $\bar{G}$ as a union of some subgraphs of $\bar{G}$, then $M(G)$ is isomorphic to the submonoid of $M(\overline{G_1})\times\cdots\times M(\overline{G_n})$, generated by the elements of the form $\iota(a)=(a_1,a_2,\ldots,a_n)$, where $a\in A$ and a_i is a if $a\in G_i$ and 1 otherwise.*

(3) *Let $M(G)$ be a subdirect product of M_1 and M_2. Then $\bar{G}=\bar{H}\cup\bar{K}$ and $M(H)\cong M_1$ and $M(K)\cong M_2$.*

(4) *Let $M(G)$ be an amalgamation of $M_1 * M_2$ by some common submonoid M_3. Then $G=H\cup K$ and $M(H)\cong M_1$, $M(K)\cong M_2$, $M(H\cap K)\cong M_3$.*

Proof. Since the proofs of (1) and (2) are only transcriptions of the proofs of Theorems 2.5 and 2.6, we prove only (3) and (4) for arbitrary graphs $G=(A,\theta)$.

(3) Let $R=\{(l_a,r_a)\mid a\in A\}$ the subset of $M_1\times M_2$ corresponding to A. The sets $A_1=\{a\in A\mid l_a\neq 1\}$ and $A_2=\{a\in A\mid r_a\neq 1\}$ need not be disjoint, but $A_1\cup A_2=A$. If $a\in A_1$, $b\in A_1-A_2$, then $\{a,b\}\in\theta\Leftrightarrow l_a\cdot l_b=l_b\cdot l_a$. Similarly, if $a\in A_2$, $b\in A_2-A_1$, then $a,b\in\theta\Leftrightarrow r_a\cdot r_b=r_b\cdot r_a$. If $a,b\in A_1\cap A_2$, then $a,b\in\theta\Leftrightarrow(l_a,r_a)\cdot(l_b,r_b)=(l_b,r_b)\cdot(l_a,r_a)$. Finally, for $a\in A_1-A_2$, $b\in A_2-A_1$, we have $(a,b)\in\theta$. Hence for $H=(A_1,\theta)$ and $K=(A_2,\theta)$ we find $\bar{G}=\bar{H}\cup\bar{K}$ and $M(H)$ is a submonoid of M_1 and $M(K)$ is a submonoid of M_2. Now we obtain $M(H)\cong M_1$ and $M(K)\cong M_2$, because $M(G)$ is a subdirect product of M_1 and M_2.

(4) Let us denote the classes of $M_1 * M_2$ amalgamated by M_3 by the symbols $[w]$ with $w\in A^*$. Since A is a minimal generating system for $M(G)$, the corresponding set $\{[a]\mid a\in A\}$ is a minimal generating system, and for $a,b\in A$, $a\neq b$, $[a]\cdot[b]=[b]\cdot[a]$ holds if and only if $\{a,b\}\in\theta$. For the subsets $A_1=\{a\mid[a]\cap M_1\neq\emptyset\}$ and $A_2=\{a\mid[a]\cap M_2\neq\emptyset\}$ of A we obtain $G=(A_1,\theta)\cup(A_2,\theta)$ and it follows readily that $M_1\cong M(A_1,\theta)$, $M_2\cong M(A_2,\theta)$, and $M_3\cong M(A_1\cap A_2,\theta)$. □

From this theorem we see that in Theorems 2.5 and 2.6, it is sufficient to take a minimal cover of the edges of G by (maximal) cliques and a minimal cover of $\bar{G}$ by (maximal) cliques, respectively. By a *cover by cliques* of the graph $G=(A,\theta)$ we mean a cover $A=C_1\cup C_2\cup\cdots\cup C_n$ of A such that $\theta=\binom{C_1}{2}\cup\binom{C_2}{2}\cup\cdots\cup\binom{C_n}{2}$. It is called *minimal*, if the number n is minimal. Note that this implies $C_1,C_2,\ldots,C_n$ to be *maximal* cliques. We call the number n the *independence number* $\beta(G)$ of G. The canonical form of a graph is a particular cover by cliques not being minimal in general. Abusively we call a cover of the graph $\bar{G}$ by cliques (of $\bar{G}$) a *cover of G by anticliques*. Taking a minimal cover of G by anticliques we obtain the corresponding representation of $M(G)$ as a subdirect product of free monoids with minimal redundancy according to Theorem 2.7(2). Considering our pipeline-model of Section 4, the number of anticliques in such a minimal cover of G by anticliques might be called the *parallelity number* $\pi(G)$ of the graph G since it is the minimal

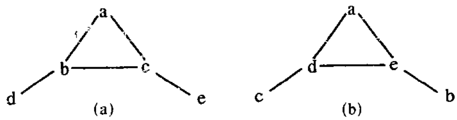


Fig. 3.

number of pipelines necessary to represent the elements of $M(G)$. On the other hand we may use representations of G with higher redundancy to obtain representations of $M(G)$ with additional useful properties.

The numbers $\beta(G)$ and $\pi(G)$ should not be confused with the so-called clique cover number and chromatic number, respectively, whose definition does not take into account the covering of the edges of the given graph but only the vertices. Obviously every cover by cliques of the edges is also a cover of the vertices of a graph, hence the clique cover number is bounded above by $\beta(G)$, and the chromatic number by $\pi(G)$.

Corollary 2.8 (Cori and Perrin [4], Duboc [6]). *Let $G=(A,\theta)$ be a graph and $u, v \in A^*$. For $B \subseteq A$ let π_B be the projection of A^* onto B^* and for $a, b \in A$ denote $\pi_{\{a\}}$ by π_a and $\pi_{\{a,b\}}$ by $\pi_{a,b}$. Then $u=v$ in $M(G)$ iff for all $a, b \in \theta$: $\pi_{a,b}(u)=\pi_{a,b}(v)$ and for all $a \in B = A - \bigcup \bar{\theta}$: $\pi_a(u)=\pi_a(v)$.*

Proof. Apply Theorem 2.7 to the decomposition $\bar{G}=\bigcup_{\{a,b\}\in\bar{\theta}} C_{\{a,b\}} \cup \bigcup_{a\in B}(a,\emptyset)$. □

Proposition 2.9. *Let G and H be graphs. If $M(G)\cong M(H)$, then $G\cong H$.*

Proof. This is a consequence of Theorem 2.6 together with Theorem 2.7(3), for example. □

To illustrate Theorems 2.5 and 2.6 we consider the following example. Let $A=\{a, b, c, d, e\}$ and the graph Q be given as shown in Fig. 3a. Then the graph $\bar{Q}$ is as shown in Fig. 3b, and we have the maximal cliques $\{a, b, c\}$, $\{b, d\}$, and $\{c, e\}$ of Q. Therefore $M(Q)$ is isomorphic to $\{a, b_1, c_1\}^{\oplus} * \{b_2, d\}^{\oplus} * \{c_2, e\}^{\oplus} / b_1=b_2$, $c_1=c_2$. According to the maximal cliques $\{a, d, e\}$, $\{c, d\}$, and $\{b, e\}$ of $\bar{Q}$ we obtain: $M(Q)$ is isomorphic to the submonoid of $\{a, d, e\}^* \times \{c, d\}^* \times \{b, e\}^*$, generated by the elements $(a, 1, 1)$, $(1, 1, b)$, $(1, c, 1)$, $(d, d, 1)$, and $(e, 1, e)$.

We denote the skeleton of a graph $G=(A,\theta)$ by the symbol $\mathscr{S}(G)$. The set of cliques of G is a $\cap$-sub-semilattice 2^G of 2^A, considered as a semilattice. We shall see that the clique-semilattice of $\mathscr{S}(G)$ characterizes G in a certain sense.

Lemma 2.10. *Let G and H be graphs. Then*

(1) *$2^{\mathscr{S}(G)}$ is isomorphic to a sub-semilattice of 2^G;*

(2) *$\mathscr{S}(G\oplus H)=\mathscr{S}(G)\oplus\mathscr{S}(H)$;*

(3) *$\mathscr{S}(G\otimes H)=\mathscr{S}(\mathscr{S}(G)\otimes\mathscr{S}(H))$;*

(4) *$\mathscr{S}(G)=\mathscr{S}(\mathscr{S}(G))$.*

Proof. (1) $\mathscr{S}(G)$ may be identified with the set of atoms ($=\cap$-irreducibles) in $2^{\mathscr{S}(G)}$. The embedding of $2^{\mathscr{S}(G)}$ into 2^G is accomplished by the mapping $[a]\mapsto\{b\mid b\in[a]\}$.

(2) The definition of the canonical equivalence can be reformulated

$$(a,b)\in\eta \Leftrightarrow \text{the star of } a \text{ equals the star of } b,$$

where the star of a is the set $\{a\}\cup\{c\in A\mid\{a,c\}\in\theta\}=\{a\}\cup A_a$. Therefore, in $G\oplus H$, no vertex of G can be equivalent to a vertex of H, from where (2) follows immediately.

(3) It is easy to see that points a and b have the same star in $G\otimes H$ if and only if $[a]$ and $[b]$ have the same star in $\mathscr{S}(G)\otimes\mathscr{S}(H)$.

(4) In $\mathscr{S}(G)$ two points are equivalent iff they are equal. □

We come back to the algebraic structure of the set $\mathscr{G}$ of all finite graphs. Since $\otimes$ can be uniquely expressed by $\oplus$ and $^-$, $\mathscr{G}$ can be considered as the free $(\oplus,\otimes,\Phi)$-algebra and the free $(\oplus,{}^-,\Phi)$-algebra as well. Note that in the second case the minimal generating system for this algebra is about half the size of the minimal generating system of the $(\oplus,\otimes,\Phi)$-algebra $\mathscr{G}$, because we only need one of G and $\bar{G}$ for every graph G in the minimal generating system of $\mathscr{G}$. Every $(\oplus,{}^-,\Phi)$-subalgebra of $\mathscr{G}$ is also a $(\oplus,\otimes,\Phi)$-subalgebra, but not vice versa: e.g. the $(\oplus,\otimes,\Phi)$-algebra generated by a single graph which is not isomorphic to its complementary graph is not a $(\oplus,{}^-,\Phi)$-algebra. On the other hand let $\mathscr{B}$ be a subset of $\mathscr{G}$ with the property $G\in\mathscr{B}\Rightarrow\bar{G}\in\mathscr{B}$. Then the $(\oplus,{}^-,\Phi)$-subalgebra of $\mathscr{G}$ and the $(\oplus,\otimes,\Phi)$-subalgebra of $\mathscr{G}$ generated by $\mathscr{B}$ coincide. In particular the set $\mathscr{P}$ of all primitive graphs and the set $\mathscr{I}$ of graphs generated by those graphs whose complement is isomorphic to itself are $(\oplus,{}^-,\Phi)$-subalgebras of $\mathscr{G}$ with $\Phi\subseteq\mathscr{P}\subseteq\mathscr{I}\subseteq\mathscr{G}$. Note that every graph with isomorphic complement must be indecomposable.

3. The clique-polynomial of a graph

As usual we denote, for a given locally finite monoid M, the monoid-ring over $\mathbf{Z}$ by $\mathbf{Z}\langle\langle M\rangle\rangle$, and $\mathbf{Z}\langle\langle A^{\oplus}\rangle\rangle$ by $\mathbf{Z}[[A]]$. $\mathbf{Z}\langle M\rangle$ and $\mathbf{Z}[A]$ are the corresponding rings of polynomials.

Let $G=(A,\theta)$ be a graph and $M=M(G)$. By definition, the clique-polynomial of G is the polynomial $p(G)\in\mathbf{Z}\langle M\rangle$

$$p(G)=\sum(-1)^n a_1a_2\ldots a_n,$$

where the sum is taken over all cliques $\{a_1,a_2,\ldots,a_n\}$ of G, and $p(\Phi)=1$. Observe that the empty set is always a clique. It is clear that the definition of $p(G)$ is independent of the order of $a_1,a_2,\ldots,a_n$ for every clique $\{a_1,a_2,\ldots,a_n\}$. It is obvious that the mapping $G\mapsto p(G)$ is an injection from the class of all graphs into the class of all polynomials. In this section we collect some properties of the clique-polynomial.

The graph G, consisting of a single point a, has the clique-polynomial $p(G)=1-a$. If G and H are graphs, then every non-empty clique of $G\oplus H$ is a clique of either G or H and conversely. It follows that

$$p(G\oplus H)-1=(p(G)-1)+(p(H)-1). \tag{10}$$

The union of a clique of G with a clique of H is a clique of $G\otimes H$. Moreover, every clique of $G\otimes H$ is a disjoint union of a clique in G and a clique in H which are both uniquely determined. Since parities behave correctly, and the order of the letters within cliques is immaterial, we obtain

$$p(G\otimes H)=p(G)\cdot p(H). \tag{11}$$

From these formulae we obtain as special cases the clique-polynomials for the clique and the anticlique on a set $A=\{a_1,a_2,\ldots,a_n\}$

$$p(a_1\oplus a_2\oplus\cdots\oplus a_n)=1-\sum_{a\in A} a, \tag{12}$$

$$p(a_1\otimes a_2\otimes\cdots\otimes a_n)=\prod_{a\in A}(1-a). \tag{13}$$

We say that a finite family $(G_i)_{i\in I}$ of graphs has the *clique-property* (CP) if every (maximal) clique of the graph $\bigcup_{i\in I} G_i$ is a (maximal) clique of at least one of the graphs G_i. A particular family having (CP) is the set of all maximal cliques of a given graph. We always assume I to be totally ordered.

Proposition 3.1. *Let $(G_i)_{i\in I}$ be a finite family of graphs with (CP). Then*

$$\sum_{J\subseteq I}(-1)^{|J|}p\left(\bigcap_{i\in J} G_i\right)=0.$$

In particular, we have for $G=\bigcup_{i\in I} G_i$

$$p(G)=\sum_{\emptyset\neq J\subseteq I}(-1)^{|J|-1}p\left(\bigcap_{i\in J} G_i\right). \tag{14}$$

If $G=C_1\cup C_2\cup\cdots\cup C_n$ is the canonical form of G and C_J is the set of vertices of $\bigcap_{i\in J} C_i$ then, for $I=\{1,\ldots,n\}$

$$p(G)=\sum_{\emptyset\neq J\subseteq I}(-1)^{|J|-1}\prod_{c\in C_J}(1-c). \tag{15}$$

Let $f:G\to\mathcal{S}(G)$ be the natural graph-morphism from G onto the skeleton of G. Then $p(G)=\sigma_f(p(\mathcal{S}(G)))$, where σ_f is the substitution $\sigma_f(b)=1-\prod_{f(a)=b}(1-a)$ for every arbitrary vertex b of $\mathcal{S}(G)$. Let $f:\bar{G}\to\mathcal{S}(\bar{G})$ be the natural graph-morphism from $\bar{G}$ onto the skeleton of $\bar{G}$. Then $p(G)=\overline{\sigma_f}(p(\overline{\mathcal{S}(\bar{G})}))$, where $\overline{\sigma_f}$ is the substitution $\overline{\sigma_f}(b)=\prod_{f(a)=b} a$ for every arbitrary vertex b of $\mathcal{S}(G)$.

Proof. For every clique C of $\bigcup_{i\in I} G_i$ set $J(C)=\{i\in I \mid C$ is a clique in $G_i\}$. From (CP), it follows that $J(C)\neq\emptyset$; thus $j(C):=\min J(C)$ always exists. We consider the set $\mathscr{F}=\{(C,J)\mid C$ is a clique of $\bigcup_{i\in I} G_i$, $J\subseteq J(C)\}$ and define the following mapping α from $\mathscr{F}$ into itself:

$$\alpha(C,J)=\begin{cases}(C,J-\{j(C)\}) & \text{if } j(C)\in J,\\ (C,J\cup\{j(C)\}) & \text{if } j(C)\notin J.\end{cases}$$

The mapping α is an involution on $\mathscr{F}$ without fixed points, hence it is a bijection between the set of all cliques of $\bigoplus_{J\subseteq I,|J|\text{even}}(\bigcap_{i\in J} G_i)$ and the set of all cliques of $\bigoplus_{J\subseteq I,|J|\text{odd}}(\bigcap_{i\in J} G_i)$. Hence these sets can be considered to be identical and since α is sign-reversing, we obtain the first equality.

The second equality follows from the first one if we note that $\bigcap_{i\in\emptyset} G_i=\bigcup_{i\in I} G_i=G$.

If every G_i is a clique, then $\bigcap_{i\in J} G_i$ is a clique for every non-empty subset J of I, and, as we saw before, $p(\bigcap_{i\in J} G_i)=\prod_{c\in C_J}(1-c)$, hence equality (15) follows from (14) and (12, 13). The natural morphism from G onto its skeleton $\mathscr{S}(G)$ has the property that the inverse image of every vertex b of $\mathscr{S}(G)$ is a clique of G which has $\prod_{f(a)=b}(1-a)$ as clique-polynomial. Maximal cliques of $\mathscr{S}(G)$ containing b are in 1-1-correspondence to maximal cliques of G containing some vertex a with $f(a)=b$. Since σ_f performs exactly this substitution, we obtain $p(G)=\sigma_f(p(\mathscr{S}(G)))$.

For the natural morphism from $\bar{G}$ onto its skeleton $\mathscr{S}(\bar{G})$, the inverse image of every vertex b of $\mathscr{S}(\bar{G})$ is a clique of $\bar{G}$, hence an anticlique of G which has the clique-polynomial $1-\sum_{f(a)=b} a$. This means to substitute $\sum_{f(a)=b} a$ for b. □

The proposition, together with equalities (10) and (11), says that every clique-polynomial can be generated from clique-polynomials of some indecomposable and skeletal graphs by addition, multiplication, and substitution.

The following example gives some applications of these formulae. Let the following graph L_4 be given:

$$a \text{ --- } b \text{ --- } c \text{ --- } d.$$

We can write $L_4=G\cup H$ for

$$G:\quad a \text{ --- } b \text{ --- } c \quad\text{and}\quad H:\quad b \text{ --- } c \text{ --- } d.$$

Then $G=(a\oplus c)\otimes b$, $H=(b\oplus d)\otimes c$, and the family (G,H) has (CP). We apply (10) and (11) to G and H to obtain $p(G)=(1-a-c)(1-b)$ and $p(H)=(1-b-d)(1-c)$. Now we can apply (14) and obtain

$$\begin{aligned}p(L_4)&=p(G)+p(H)-p(G\cap H)\\&=(1-a-c)(1-b)+(1-b-d)(1-c)-(1-b)(1-c).\end{aligned}$$

The canonical form for L_4 is $L_4=(a\otimes b)\cup(b\otimes c)\cup(d\otimes c)$. According to (15) we end up with

$$\begin{aligned}p(L_4)&=(1-a)(1-b)+(1-c)(1-b)+(1-d)(1-c)-(1-b)-(1-c)-1+1\\&=1-a-b-c-d+ab+cb+dc.\end{aligned}$$

To see that property (CP) is necessary, consider the decomposition $C_{\{a,b,c\}}=(a\otimes b)\cup(b\otimes c)\cup(a\otimes c)$. Applying (14) to this expression yields

$$\begin{aligned}
&p((a\otimes b)\cup(b\otimes c)\cup(a\otimes c))\\
&\quad=p((a\otimes b)\cup(b\otimes c))+p(a\otimes c)-p(a\oplus c)\\
&\quad=p(G)+p(a\otimes c)-p(a\oplus c)\\
&\quad=(1-a-c)(1-b)+(1-a)(1-c)-(1-a)-(1-c).
\end{aligned}$$

This expression cannot be equal to $p(C_{\{a,b,c\}})=p(a\otimes b\otimes c)=(1-a)(1-b)(1-c)$ since the degree of $p(C_{\{a,b,c\}})$ is 3, but the degree of $p((a\otimes b)\cup(b\otimes c)\cup(a\otimes c))$ is 2.

The graph G above is the skeleton of the graph K, shown in Fig. 4. Applying the substitution σ_f for the graph-morphism $f:K\to G$ with $f(a)=a$, $f(b)=f(d)=b$, $f(c)=c$, we obtain from $p(G)=(1-a-c)(1-b)$: $p(K)=(1-a-c)(1-(1-(1-b)(1-d)))=(1-a-c)(1-b)(1-d)$.

K:

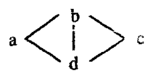


Fig. 4.

On the other hand the graph $a\oplus b\oplus d$ is the skeleton of $\bar{K}$, hence by the last part of Proposition 3.1 we can obtain $p(G)$ by applying the substitution $a\mapsto a+c$ to the clique-polynomial of $\mathcal{S}(\bar{K})=a\otimes b\otimes d$. This also results in $p(K)=(1-a-c)(1-b)(1-d)$.

For every graph $G=(A,\theta)$ we define a partial order on A to be an order on the graph G if every clique of G is a chain in the order, i.e. a totally ordered subset of A. Of course, it is sufficient that every maximal clique of G is a chain. We call G, together with such an order, an *ordered graph*. Every graph G can be ordered, because, for example, with every total order on the set of vertices, every graph is an ordered graph. For every ordered graph $G=(A,\theta)$ we can represent the polynomial $p(G)$ by a polynomial in $\mathbf{Z}\langle A^*\rangle$ by ordering every monomial of $p(G)$ according to the partial order on A. We forget about the order of G, if we consider $p(G)$ as an element of $\mathbf{Z}\langle M(G)\rangle$, or $\mathbf{Z}[A]$, because then the order of the letters within every clique is immaterial. Of course, every order on G induces an order on every connected component of G, and conversely, the disjoint union of the orders on the components is an order on G. Hence (10) is also valid for ordered graphs. If $G=(A,\theta)$ and $H=(B,\phi)$ are ordered, then $G\otimes H$ is ordered by the order on the disjoint union of A and B, which extends the given orders by $a<b$ if $a\in A$ and $b\in B$. Now, also (11) is valid for ordered graphs, but $p(G\otimes H)\neq p(H\otimes G)$, whereas $p(G\oplus H)=p(H\oplus G)$. Note that, if G is an ordered graph, then $\mathcal{S}(G)$ is also an

ordered graph by defining the following derived order on $\mathscr{S}(G)$:

$$[a]<[b] \Leftrightarrow$$
$$[a]\neq[b] \text{ and } a'<b' \text{ holds for the minimal elements } a' \text{ of } [a] \text{ and } b' \text{ of } [b].$$

Conversely, if $\mathscr{S}(G)$ is an ordered graph, then G can be ordered by taking an arbitrary fixed total order on every clique $[a]$ of $G(a\in A)$ and extending this by

$$a<b \Leftrightarrow a\neq b \quad\text{and}\quad [a]<[b].$$

In this case the natural graph-morphism $f: G\to\mathscr{S}(G)$ is a morphism of ordered graphs (i.e. $a\leqslant b\Rightarrow f(a)\leqslant f(b)$). The two processes are not the reverse of each other. Take the order $d<a<b$, $d<c<b$, $d<b$ on the graph K (Fig. 4). Going to $\mathscr{S}(K)$ and back to K will never produce the original order on K. It also follows that a labelling of an ordered graph cannot be considered as a homomorphism of ordered graphs in general.

Part of the following theorem grew out of some fruitful discussions with C. Reutenauer. We recall that a monoid M is said to be irreducible if $M\cong M_1\times M_2$ implies M_1 or M_2 to be trivial.

Theorem 3.2. *For every ordered graph $G=(A,\theta)$, the following assertions are equivalent*:

(1) *$p(G)$ is irreducible over $\mathbb{Z}[A]$*;
(2) *$p(G)$ is irreducible over $\mathbb{Z}\langle M(G)\rangle$*;
(3) *$p(G)$ is irreducible over $\mathbb{Z}\langle A^*\rangle$*;
(4) *$\bar{G}$ is connected*;
(5) *$M(G)$ is irreducible.*

Proof. (1)$\Rightarrow$(2) and (2)$\Rightarrow$(3) are obvious.

(3)$\Rightarrow$(4): If $\bar{G}$ is not connected, then we can find subgraphs $\bar{H}$ and $\bar{K}$ of $\bar{G}$ such that $\bar{G}=\bar{H}\oplus\bar{K}$. Then $G=H\otimes K$ and from (11) we conclude $p(G)=p(H)\cdot p(K)$.

(4)$\Rightarrow$(1): We assume $\bar{G}$ to be connected, but $p(G)=h\cdot k$ is a factorization over $\mathbb{Z}[A]$ and h and k are not constant. We denote by $\deg_a(p)$ the partial degree of a polynomial $p\in\mathbb{Z}\langle M(G)\rangle$ in the symbol $a\in A$. For every symbol $a\in A$ we have $\deg_a(p(G))=1$ and from

$$p=h\cdot k\Rightarrow\forall a\in A\quad \deg_a(p)=\deg_a(h)+\deg_a(k)$$

for all $p, h, k\in\mathbb{Z}[A]$, it follows that

$$\forall a\in A\quad \deg_a(h)=1 \Leftrightarrow \deg_a(k)=0. \tag{16}$$

This means that a symbol occurs in h iff it does not occur in k. Since every symbol $a\in A$ occurs as a monomial in $p(G)$ we infer that every $a\in A$ occurs as a monomial in either h or k, but not in both. On the other hand h must have at least one symbol $a\in A$ as a monomial, for otherwise $\deg_a(k)=1$ for every $a\in A$, hence by (16), h would be constant. Similarly, k has at least such a symbol $b\in A$.

Now $\bar{G}$ is connected, hence we can find an edge in $\bar{\theta}$ joining some symbols a and b, occurring as monomials in h and k, respectively. Then we know that the coefficient of ab in $p(G)$ is zero: $\langle p(G), ab\rangle = 0$. But

$$\langle p(G), ab\rangle = \langle h, 1\rangle \cdot \langle k, ab\rangle + \langle h, a\rangle \cdot \langle k, b\rangle + \langle h, b\rangle \cdot \langle k, a\rangle + \langle h, ab\rangle \cdot \langle k, 1\rangle,$$

and we know $\langle h, a\rangle \cdot \langle k, b\rangle + \langle h, b\rangle \cdot \langle k, a\rangle = 1$. The coefficient $\langle h, ab\rangle$ cannot be different from 0, since otherwise $\deg_b(h) = 1$ and hence $\deg_b(k) = 0$, contradicting our choice of b. Similarly, $\langle k, ab\rangle = 0$, and the result of the above sum is 1, which is a contradiction.

The equivalence of (4) of (5) was already shown in Lemma 2.1. □

Corollary 3.3. (1) *Let F be an arbitrary field and $p_F(G)$ the image of $p(G)$ under the morphism $\mathbf{Z}[A] \to F[A]$ induced by the ring-morphism $z \mapsto z \cdot 1$ from $\mathbf{Z}$ to F. Then $p_F(G)$ is irreducible iff $\bar{G}$ is connected.*

(2) *$p(G)$ is irreducible over $\mathbf{Z}\langle M(G)\rangle$ iff $p(\bar{G}) + 1$ cannot be written as the sum of two non-trivial clique-polynomials.*

(3) $p(G) = p(H) \cdot p(K) \Leftrightarrow p(\bar{G}) = p(\bar{H}) + p(\bar{K}) - 1$.

Proof. (1) Every factorization of $p(G)$ gives a factorization of $p_F(G)$. For the opposite direction, note that the proof of the theorem does not make use of any ring-element different from 0 or 1.

(2) and (3) If $p(G) = h \cdot k$ is a factorization over $\mathbf{Z}\langle M(G)\rangle$, then $\bar{G} = \bar{H} \oplus \bar{K}$ by Theorem 3.2 and $p(\bar{G}) = p(\bar{H}) + p(\bar{K}) - 1$ by equation (10). For the opposite direction let $p(\bar{G}) = p(\bar{H}) + p(\bar{K}) - 1$ for some arbitrary graphs $\bar{H}$ and $\bar{K}$. From this we can derive $\bar{G} = \bar{H} \oplus \bar{K}$, hence $G = H \otimes K$ and $p(G) = p(H) \cdot p(K)$. □

4. Cross-sections for free partially commutative monoids

In Section 2 we have seen several different presentations for free partially commutative monoids $M(A, \theta) \cong A^*/\rho$, where $\rho = \{ab = ba \mid \{a, b\} \in \theta\}$. In this section we derive normal forms for the elements of $M(A, \theta)$, i.e. for the ρ-classes as subsets of A^*, by using the different presentations.

A subset T of A^* is called a *cross-section* or a *transversal* for $M(A, \theta)$ if T contains exactly one element of every ρ-class. T is called *rational*, if T is a member of the class $\mathrm{Rat}(A^*)$ of subsets of A^*, where for some arbitrary monoid M the class $\mathrm{Rat}(M)$ of all rational subsets of M is defined by the following scheme:

- $\emptyset \in \mathrm{Rat}(M)$, $\{m\} \in \mathrm{Rat}(M)$ for every $m \in M$,
- $U, V \in \mathrm{Rat}(M) \Rightarrow U \cdot V = \{uv \mid u \in U, v \in V\} \in \mathrm{Rat}(M)$ and $U \cup V \in \mathrm{Rat}(M)$,
- $U \in \mathrm{Rat}(M) \Rightarrow U^* = \bigcup_{n \geq 0} U^n \in \mathrm{Rat}(M)$,

and $U^0 = \{1\}$, $U^{n+1} = U \cdot U^n$, and $\emptyset^* = \{1\}$. If some cross-section T and $w \in A^*$ are given, then we can consider the single element in the intersection $T \cap [w]_\rho$ as the representative for w. Sometimes it is convenient to consider T as a mapping associating to every $w \in A^*$, the representative $T(w)$.

Let G be an ordered graph with canonical form $G = C_1 \cup C_2 \cup \cdots \cup C_n$. According to Theorem 2.5, $M(G)$ can be considered as the free product of the monoids $C_1^{\oplus}, C_2^{\oplus}, \ldots, C_n^{\oplus}$, amalgamated by the monoids $C_{i,j}^{\oplus}$, where $C_{i,j} = C_i \cap C_j$ for $1 \leq i < j \leq n$. This means that we are free to consider elements of $C_{i,j}^{\oplus}$ as belonging to $C_i^{\oplus}$ or $C_j^{\oplus}$. To obtain a representative of $[w]$ for some given $w \in A^*$ we write $[w] = [w_1] \cdot [w_2]$, such that $w_1 \in C^*$ for some $C \in \{C_1, \ldots, C_n\}$, every letter of w_1 occurs only once, and w_1 is as long as possible. From Theorem 2.5, it follows that this factorization is uniquely determined, hence also the set of all the letters occurring in w_1. If we order the letters of w_1 according to the order on G, which is total on every clique, we obtain a word $u_1 \in A^*$. u_1 is the first factor of the representative for w. If we define the representative of 1 to be 1 and assume by induction that the element $[w_2]$ already has a unique representative $u_2 \cdots u_t$ then $F(w) = u_1 \ldots u_t \in A^*$ is the unique representative for w. $F(w)$ is called Foata's *normal form* of w.

The original definition of Foata's normal form [2] uses a total order on the given graph. Formally our definition is slightly more general, but every total order extending the given order on the graph defines the same Foata's normal form which coincides with our definition of Foata's normal form.

Let us consider the following example. As before, let Q be as shown in Fig. 5 and $a < b < c$, $e < c$, $b < d$. Then Q is an ordered graph and Foata's normal form of the word $w = baeacbcedacdb \in A^*$ can be computed in the following way. The longest left factor of $[w]$ which contains only one letter from some maximal clique C is $w_1 = bac$, because e does not commute with any of a, b, and d, and thus $[w] = [w_1] \cdot [ceabedacdb]$. The product of all the letters occurring in w_1 is, in the given order on G, $u_1 = abc$, and $[w] = [abc] \cdot [ceabedacdb]$. The longest left factor of $[ceabedacdb]$ containing only letters from one clique is $[ce]$, hence $u_2 = ec$, and $[w] = [abc] \cdot [ec] \cdot [abedacdb]$. Proceeding inductively in this way we obtain $F(w) = abc|ec|ab|e|bd|ac|d$, where we indicate the factorization by vertical strokes.

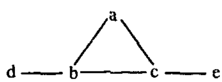


Fig. 5.

Foata's normal form $F(w) = u_1 \ldots u_t$ for some $w \in A^*$ can be equivalently described by the following conditions:

- every u_i is a product of letters in strictly ascending order, contained in some $C \in \{C_1, \ldots, C_n\}$,
- for every letter a in u_{i+1} there is a letter b in u_i such that $\{a, b\} \notin \theta$,
- $F(w) \in [w]$.

From this description it is easily seen by the following considerations that the set $T_F(G)$ of all Foata normal forms of elements of $M(G)$ relative to the given order of G is a rational subset of A^*.

Consider the set $\mathscr{C} = \{C \mid \exists 1 \leq i \leq n \ C \subseteq C_i\}$ as a set of letters and define a monoid-morphism $\varphi : \mathscr{C}^* \to A^*$ by $\varphi(C) = a_1 \ldots a_k$ for every $C \in \mathscr{C}$ such that $C = \{a_1, \ldots, a_k\}$ and $a_1 < \cdots < a_k$. Denote $K = \{CC' \mid C, C' \in \mathscr{C}, \exists c \in C' - C \text{ s.t. } C \cup \{c\} \in \mathscr{C}\}$; then $T_F(G) = \varphi(\mathscr{C}^* - \mathscr{C}^* K \mathscr{C}^*)$ is a rational subset of A^* since by Kleene's theorem $\mathrm{Rat}(\mathscr{C}^*)$ is a Boolean algebra, and homomorphisms preserve rationality.

A second normal form for the elements of a free partially commutative monoid $M(G)$ is the following *alphabetic normal form*. Similarly as for Foata's normal form, the alphabetic normal form is originally defined for some given total order $\leq$ on the graph G [3, 6]. All elements within some equivalence class $[w]$ have equal length, thus one can define for $w \in A^*$, the alphabetic normal form $L(w)$ of w as the lexicographic smallest element in the equivalence class $[w]$. Now let $G = (A, \theta)$ be an arbitrarily ordered graph and $w \in A^*$. The next lemma shows that the set of all lexicographic minimal elements in $[w]$ relative to this partial order is a one-element set.

Lemma 4.1. *Let $w, z \in A^*$ with $[w] = [z]$ and w minimal in $[w]$. Then $w \neq z \Rightarrow w < z$.*

Proof. Since $w \neq z$, we can find a longest common prefix x of w and z. Then $w = xav_1$, $z = xbv_2$, and $a, b \in A$ with $a \neq b$. b must occur in v_1, hence there is a shortest word v such that $w = xavby$. Since w and z are equivalent, b must commute with every letter in v and with a, and we see that v is contained in A_b^*, $\{a, b\} \in \theta$. Since w is minimal in $[w]$ we obtain $a < b$. $\square$

From this lemma we can see that for every $w \in A^*$ the θ-class $[w]$ contains a uniquely determined lexicographic minimal element $L(w)$, the alphabetic normal form of w. It also allows us to derive the following descriptions of $T_L(G) = \{L(w) \mid w \in A^*\}$ for $G = (A, \theta)$:

$$w \in T_L(G) \Leftrightarrow (w = xavby, x, v, y \in A^*, \{a, b\} \in \theta, a > b \Rightarrow v \notin A_b^*);$$

$$T_L(G) = A^* - \bigcup_{a > b, \{a,b\} \in \theta} A^* a A_b^* b A^*,$$

which shows that $T_L(G)$ is a rational cross-section for $M(G)$. In our last example we obtain as the alphabetic normal form of $w = baeacbcedacdb$, the word $L(w) = abeabebccdacd$.

Let us recall that the monoid-morphism ι is an embedding of $M(G)$ into $C_1^* \times C_2^* \times \cdots \times C_n^*$, where $\bar{G} = C_1 \cup C_2 \cup \cdots \cup C_n$ is the canonical form or an arbitrary cover of $\bar{G}$ by cliques of $\bar{G}$. The generating system is the set of n-tuples $\iota(A) = \{\iota(a) \mid a \in A\}$ and thus for every $w \in A^*$ the n-tuple $\iota(w)$ characterizes the equivalence class $[w]$:

$$[v] = [w] \Leftrightarrow \iota(v) = \iota(w).$$

Hence for fixed $w \in A^*$, every $v \in A^*$ with the property $\iota(v) = \iota(w)$ can be considered as a representative for $[w]$. In general such a v is not uniquely determined, but every strategy producing for every $[w]$, a unique v such that $\iota(v) = \iota(w)$ gives a cross-section of $M(G)$. Let us consider again our example above, where $\tilde{Q} = (a\otimes d\otimes e)\cup(b\otimes e)\cup(c\otimes d)$ is the canonical form of $\bar{Q}$. The word $w =$ *baeacbcedacdb* has $\iota(w) = (aeaedad, bebeb, ccdcd)$. A possible strategy to obtain a unique v is: "Read $\iota(w)$ from left to right and take the first letter, which is in the first place in every component, which contains this letter". By this procedure we obtain from the n-tuple $\iota(w) = (aeaedad,\ bebeb,\ ccdcd)$

$$\begin{aligned}\iota(w) &= \iota(a)\cdot(eaedad, bebeb, ccdcd) = \iota(ab)\cdot(eaedad, ebeb, ccdcd)\\ &= \iota(abe)\cdot(aedad, beb, ccdcd) = \iota(abea)\cdot(edad, beb, ccdcd)\\ &= \iota(abeab)\cdot(edad, eb, ccdcd) = \iota(abeabe)\cdot(dad, b, ccdcd)\\ &= \iota(abeabeb)\cdot(dad, 1, ccdcd) = \iota(abeabebc)\cdot(dad, 1, cdcd)\\ &= \iota(abeabebcc)\cdot(dad, 1, dcd) = \iota(abeabebccd)\cdot(ad, 1, cd)\\ &= \iota(abeabebccda)\cdot(d, 1, cd) = \iota(abeabebccdac)\cdot(d, 1, d)\\ &= \iota(abeabebccdacd),\end{aligned}$$

and the representative for w is $v = abeabebccdacd$. We call v the *left-right normal form* of w and denote it by $\mathrm{LR}(w)$, writing frequently LR normal form.

Note that for an arbitrary graph $G = (A, \theta)$ and $w \in A^*$ the representative $\mathrm{LR}(w)$ depends on the order of the anticliques of G in $C_1^* \times C_2^* \times \cdots \times C_n^*$, but does not depend on an a priori order on G. The set of all left-right normal forms is called $T_{\mathrm{LR}}(G)$.

We can imagine the n-tuple $\iota(w)$ to be represented as a collection of pipelines, each representing one component. We start with empty pipes and begin reading in the first letter of w, in our example in Fig. 6 this is b, into every pipe which represents a component i such that $b \in C_i$. Now we can read the contents of the pipeline according to Fig. 7. We say, a letter is top-visible, if it is in top position in every pipe which contains this letter. Every top-visible letter is a left-factor of w, and vice versa, every letter, multiplied from the left, will be top-visible. Dually, bottom-visible letters are exactly the right-factors of w, which are elements of A.

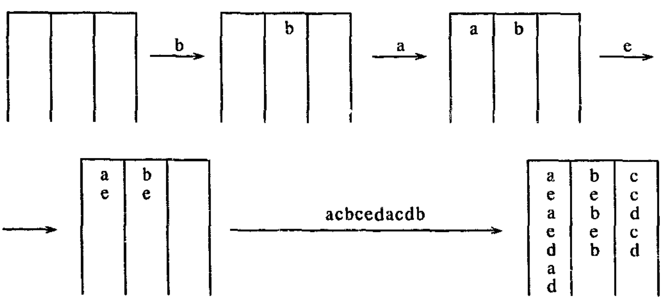


Fig. 6.

a	b	c
e	e	c
a	b	d
e	e	c
d	b	d
a		
d		

Fig. 7.

For example, in Fig. 7 the letters a, b, c are top-visible and b, d are bottom-visible. The rule "remove all the visible letters of the top slice and multiply them according to some given order of the graph to obtain some slice-product, and then multiply all the slice-products in the order of the slices" gives the Foata's normal form relative to the order of the graph. Similarly, taking at every moment the visible letter which is minimal in the order of the graph, gives the alphabetic normal form. Note that this letter is unique, because all the letters visible at the same time belong to one clique. Finally, taking always the leftmost among all visible letters produces the left-right normal form.

From this visualization of the various normal forms it is immediate that a word $w = u_1 \cdot \ldots \cdot u_t \in T_F(G)$ has a factorization $[w] = [u] \cdot [v]$ for some $u \in T_F(G)$ which contains only letters a, b such that $\{a, b\} \in \theta$ if and only if $[u]$ is a factor of $[u_1]$. Also Lemma 4.1 is now obvious since it simply states that one cannot do better than taking in every slice of visible letters the order induced by the order on the graph.

We recall that free partially commutative monoids can be considered as a model for nonsequential processes. The letters of the alphabet A are imagined as elementary actions and commutation of two letters means that the composition of the corresponding actions is independent of their order. Foata's normal form for some given $w \in A^*$, together with its factorization $F(w) = u_1|u_2|\ldots|u_t$ contains the following information:

- t is the minimal number of steps necessary to perform w if arbitrarily high parallelity is allowed;
- the necessary number of processes working in parallel to obtain the performance of w in t steps is the maximal length of a factor in Foata's normal form.

This number of processes is independent of w for those w using all the letters of the alphabet. From Theorem 2.7(2) we have seen in Section 2 that this number is the size of a minimal cover of G by anticliques and therefore this number has been called the parallelity number of the graph G. With this number of parallel processes, and processing according to Foata's normal form $F(w) = u_1|u_2|\ldots|u_t$ we have optimal performance for every $w \in A^*$.

We can also easily deduce that the set $T_{\mathrm{LR}}(G)$ is rational. Let $\bar{G} = L_1 \cup L_2 \cup \cdots \cup L_n$ be a representation of $\bar{G}$ as a union of cliques, i.e. $L_1, L_2, \ldots, L_n$ is a cover of G by anticliques. For every $a \in A$ define $i(a) = \min\{k \mid a \in L_k\}$. Then the reflexive and transitive closure of the relation "<", defined by

$$a<b \Leftrightarrow i(a)<i(b) \quad \text{and} \quad \{a,b\}\in\theta$$
$$\Leftrightarrow i(a)<i(b) \quad \text{and} \quad \forall j(1\leqslant j\leqslant n)\{a,b\}\notin L_j$$

is a partial order $\leqslant$ on the set A of vertices of G with the property that every clique of G is a chain in the order, because $a<b$ and $b<c$ imply $a\neq c$. Hence G is an ordered graph. Note that we also use the character "<" for the transitive closure of this relation. We call this partial order the *order induced by the representation* of $\bar{G}$ or induced by the representation of $M(A,\theta)$. If we take on G the order induced by the canonical form of $\bar{G}$, then the alphabetic normal form according to this order coincides with the LR normal form. This proves that Lemma 4.1 is also true for $w=\mathrm{LR}(w)$:

Property 4.2

$$w\in T_{\mathrm{LR}}(G) \Leftrightarrow (w=xavby,\ x,v,y\in A^*, \{a,b\}\in\theta,\ a>b \Rightarrow v\notin A_b^*),$$
$$T_{\mathrm{LR}}(G)=A^*-\bigcup_{a>b,\{a,b\}\in\theta} A^*aA_b^*bA^*.$$

From this characterization we can again derive that $T_{\mathrm{LR}}(G)$ is a rational subset of A^*.

Every system T of normal forms of $M(A,\theta)$ can be considered as a monoid which is isomorphic to $M(A,\theta)$ if we define the product of u and v in the monoid T as the corresponding normal form of uv. Considered as monoids, the sets T_F, T_L, and T_{LR} are thus isomorphic. Nevertheless the combinatorial structures of the sets T_F, T_L, and T_{LR} are very different. For example, every factor of some w in T_L or in T_{LR} is again in alphabetic or left-right normal form, respectively, whereas the set T_F is only closed under taking prefixes. The set T_{LR} has the particular property that the normal form of some word can be constructed without knowing the order on G explicitly, but only a canonical form of $\bar{G}$ together with some total order on the set of anticliques of G. Property 4.2 implies that the concatenation of two LR-normal forms u and w is again in LR-normal form iff for every letter a of u which commutes with some letter b of w the following is true:

$$u=xav_1,\ w=v_2by,\ a>b \Rightarrow v_1v_2\notin A_b^*.$$

From this observation we can derive the following useful results. As usual, we denote the free semigroup generated by some set A by the symbol A^+. Then $A^+=A^*-1$.

Theorem 4.3. *Let G and H be arbitrary graphs with their partial orders induced by the canonical forms of $\bar{G}$ and $\bar{H}$. Consider $G\oplus H$, $G\otimes H$ and $\mathcal{S}(G)$ as ordered graphs according to the discussion before Theorem 3.2. Then the following equalities hold:*

$$T_{\mathrm{LR}}(G\otimes H)=T_{\mathrm{LR}}(G)\cdot T_{\mathrm{LR}}(H),$$
$$T_{\mathrm{LR}}(G\oplus H)=T_{\mathrm{LR}}(G)\cdot((T_{\mathrm{LR}}(H)-1)\cdot(T_{\mathrm{LR}}(G)-1))^*\cdot T_{\mathrm{LR}}(H),$$
$$T_{\mathrm{LR}}(G)=\tau_f(T_{\mathrm{LR}}(S(G))),$$

where f is the natural morphism from G onto $S(G)$ and τ_f is the substitution $a^+ \mapsto b_1^* \dots b_n^* - 1$ with $\{b_1, \dots, b_n\} = f^{-1}(a)$ and $b_1 < \dots < b_n$. Finally, if f is the natural morphism from $\bar{G}$ onto $\mathscr{S}(\bar{G})$, and $\overline{\tau_f}$ is the substitution $a^+ \mapsto (b_1^* \dots b_n^*)^* - 1$ with $\{b_1, \dots, b_n\} = f^{-1}(a)$, then

$$T_{\mathrm{LR}}(G) = \overline{\tau_f}(T_{\mathrm{LR}}(\overline{\mathscr{S}(\bar{G})})).$$

Proof. Before establishing the inclusions in both directions, we have to check that the orders on $G \otimes H$ and on $G \oplus H$ can be considered as being induced by some total order on the anticliques of $G \otimes H$ and on $G \oplus H$, respectively. In fact, it is easy to see that for given total orders on the set of anticliques of G and H, their ordinal sum induces the desired order on $G \otimes H$ and their ordinal product induces the desired order on $G \oplus H$.

The above condition on concatenations of LR-normal forms is trivially true for each of the four right-hand sides, hence the inclusions from right to left follow. Conversely, in every word of $T_{\mathrm{LR}}(G \otimes H)$ no letter of H can be followed by a letter of G, and thus every element of $T_{\mathrm{LR}}(G \otimes H)$ splits into a product of a left-factor in $T_{\mathrm{LR}}(G)$ with a right-factor in $T_{\mathrm{LR}}(H)$.

Obviously, every element of $T_{\mathrm{LR}}(G \oplus H)$ uniquely decomposes in the desired way, because G and H are disjoint.

To see the third equality, note first that $T_{\mathrm{LR}}(G)$ is a rational expression over the set $\{a^+ \mid a \in A\}$ because Property 4.2 makes sure that for all $a \in A$, $u, v \in A^*$: $uav \in T_{\mathrm{LR}}(G) \Rightarrow ua^+v \subseteq T_{\mathrm{LR}}(G)$. Hence τ_f and $\overline{\tau_f}$ are in fact applicable substitutions and their result is not depending on the actual rational expression for $T_{\mathrm{LR}}(G)$. Now let $w \in T_{\mathrm{LR}}(G)$. w uniquely decomposes into $w = w_1 \dots w_k$ such that all the letters of w_i have the same image in $\mathscr{S}(G)$ and different w_i have different images. With Property 4.2 the third equality follows. A similar reasoning yields the last equality. $\square$

Remark 4.4. (1) For primitive graphs G we can easily derive rational expressions for $T_{\mathrm{LR}}(G)$ with the help of Theorem 4.3. The star-height of $T_{\mathrm{LR}}(G)$ is equal to $1+$the number of occurrences of $\oplus$ in a longest branch of the tree for the alternating normal form of G.

(2) Moreover, to construct rational expressions for $T_{\mathrm{LR}}(G)$ and arbitrary graphs G we see that it is sufficient to know rational expressions for $T_{\mathrm{LR}}(H)$ for all graphs H which are indecomposable and skeletal.

(3) All the rational operations used in Theorem 4.3 are unambiguously rational, hence for every primitive graph G the set $T_{\mathrm{LR}}(G)$ is unambiguously rational.

(4) Theorem 4.3 remains true if T_{LR} is replaced by T_L.

For one-point graphs $(a, \emptyset)$, the set of LR-normal forms is $a^* = 1 + a^+$. Hence Theorem 4.3, applied to the graph $G = b \otimes (a \oplus c)$ yields the rational expression

$$T_{\mathrm{LR}}(G) = (1+b^+) \cdot (1+a^+) \cdot (1+(c^+a^+)^+) \cdot (1+c^+) = b^*(a^*c^*)^*.$$

Since the complement of G has the skeleton $b \oplus d$, we have $T_{\mathrm{LR}}(\overline{S(\bar{G})}) = (1+b^+) \cdot (1+d^+)$. Applying the last equality of Theorem 4.3 for the mapping f sending a and c to d, and b to b, we obtain

$$T_{\mathrm{LR}}(G) = (1+b^+) \cdot (1+(a^*c^*)^* - 1) = b^*(a^*c^*)^*$$

again.

Occasionally we shall identify some cross-section T with the mapping of $M(G)$ into A^* having the property that the natural homomorphism of A^* onto $M(G)$ composed with T gives the identity mapping on $M(G)$. In this sense the cross-section T can be applied to the monomials of some polynomial $p \in \mathbf{Z}\langle\langle M(G)\rangle\rangle$ and defines a normal form $T(p) \in \mathbf{Z}\langle\langle A^*\rangle\rangle$ of p. The same can be done in the slightly generalized situation where A^* is replaced by an arbitrary monoid N having a surjective morphism φ onto $M(G)$. Then we may say that T is a cross-section along φ or a cross-section of $M(G)$ in N if φ is understood.

5. The Möbius-function of $M(A, \theta)$

The Möbius-function μ_M of a locally finite monoid M is defined as the inverse of the series $\zeta_M = \sum_{m \in M} m \in \mathbf{Z}\langle\langle M\rangle\rangle$: $\mu_M = \zeta_M^{-1}$. For $M = M(G)$, μ_M has been determined in [2] as $\mu_M = p(G)$. To keep this paper self-contained we reproduce a variation of Viennot's [9] elegant bijective proof of this fact.

Proposition 5.1. *For every graph G and $M = M(G)$ the Möbius-function of M is $\mu_M = p(G)$.*

Proof. We show that $p(G) \cdot \zeta_M = 1$ in $\mathbf{Z}\langle\langle M\rangle\rangle$. This is equivalent to

$$\sum (-1)^n a_1 a_2 \dots a_n \cdot m = 1,$$

where the summation is over all pairs $(\{a_1, a_2, \dots, a_n\}, m)$ with $\{a_1, a_2, \dots, a_n\}$ a clique of G and m an element of M. We fix an order on G and identify every $m \in M$ with its Foata normal form $F(m) = u_1 \cdot \ldots \cdot u_t \in T_F(G)$. To every pair $(\{a_1, a_2, \dots, a_n\}, m)$ we associate a second pair by the following procedure. Define $B = \{a \in A \mid a$ occurs in u_1 and for all $i = 1, \dots, n$ $\{a, a_i\} \in \theta\}$ and consider $B \cup \{a_1, a_2, \dots, a_n\}$. Note that this union is disjoint. This set is a clique of G, hence it has a minimal element b in the order of G. If $b \in B$, then b is a left-factor of m, hence $m = b \cdot m'$ for some $m' \in T_F(G)$, and we associate $(\{a_1, a_2, \dots, a_n, b\}, m')$; if $b \in \{a_1, a_2, \dots, a_n\}$, then we associate $(\{a_1, a_2, \dots, a_n\} - \{b\}, F(b \cdot m))$. Otherwise, the pair $(\{a_1, a_2, \dots, a_n\}, m)$ remains unchanged. This defines a sign-reversing involution α on the set of pairs (C, m), where C is a clique of G and $m \in M$, leaving fixed the pair $(\emptyset, 1)$ only, which produces 1 in the above sum. $\square$

It may happen that for the calculation of $\mu_M \cdot \zeta_M$ we do not need any commutation, i.e. if we write μ_M and ζ_M as elements of $\mathbb{Z}\langle\langle A^*\rangle\rangle$ appropriately, then $\mu_M \cdot \zeta_M = 1$ in $\mathbb{Z}\langle\langle A^*\rangle\rangle$. For example, the graph $G = b\otimes(a\oplus c)$ has $\zeta_M = \mu_M^{-1} = (1-a-c)^{-1}\cdot(1-b)^{-1}$. This can be calculated in $\mathbb{Z}\langle\langle A^*\rangle\rangle$: $\zeta_M = \sum_{n\geq 0}(a+c)^n \cdot \sum_{n\geq 0} b^n$. The support of this series is the rational cross-section $(a+c)^*b^* = (a^*c^*)^*b^*$ of $M(G)$. From the properties of $p(G)$ it follows easily that for every primitive graph G and $M = M(G)$, the product $\mu_M \cdot \zeta_M = 1$ can be calculated in $\mathbb{Z}\langle\langle A^*\rangle\rangle$. We shall see that $\mathcal{P}$ is not the maximal class of graphs with this property.

Let N be a monoid generated by A which has a surjective morphism f onto the monoid M. We say, μ_M and ζ_M *can be lifted* or have a *lifting* to $\mathbb{Z}\langle\langle N\rangle\rangle$ if we can find cross-sections T and T' of f in N such that for $\mu = T'(\mu_M)$ $(=\sum_{m\in M}\langle \mu_M, m\rangle T'(m))$ and $\zeta = T(\zeta_M)$ the relation $\mu\cdot\zeta = 1$ holds in $\mathbb{Z}\langle\langle N\rangle\rangle$. In this sense the Möbius-function of $M(A,\theta)$ may be used to calculate cross-sections in N.

After several discussions with V. Diekert there is some evidence for the following conjecture.

Conjecture 5.2. *Let $G=(A,\theta)$ be an ordered graph, $\phi = \{\{a,b\},\{b,c\}\in\theta \mid \{a,c\}\notin\theta$ and $a<b<c\}$. Then $\mu_M(G)$ and $\zeta_M(G)$ can be lifted to $\mathbb{Z}\langle\langle M(A,\phi)\rangle\rangle$.*

We call an order on a graph $G=(A,\theta)$ *perfect*, if it has the property

$$\{a,b\}\in\theta \Leftrightarrow a<b \text{ or } a>b.$$

The graph G is said to have a perfect order, if there is some order on G for which the above condition is true. The next lemma implies that every primitive graph has a perfect order. The graphs L_4 and Q from Section 2 are examples for graphs having perfect orders without being primitive.

Lemma 5.3. *Let G and H be graphs.*

(1) *$G\oplus H$ has a perfect order iff G and H have a perfect order.*
(2) *$G\otimes H$ has a perfect order iff G and H have a perfect order.*
(3) *G has a perfect order if and only if the skeleton $\mathcal{S}(G)$ has a perfect order.*
(4) *Let G and H be graphs with $\mathcal{S}(G)=\mathcal{S}(H)$. Then G has a perfect order iff H has a perfect order.*
(5) *G has a perfect order if and only if $\overline{\mathcal{S}(\bar{G})}$ has a perfect order.*

Proof. (1) and (2) are clear.

(3) Every η-class U is a clique of G and for η-classes U and V we have $(U,\binom{U}{2})\oplus(V,\binom{V}{2})$ or $(U,\binom{U}{2})\otimes(V,\binom{V}{2})$ as a subgraph of G. If G has a perfect order, then the order on $\mathcal{S}(G)$ inherited from G by $a<b\Leftrightarrow[a]<[b]$ is again perfect. If $\mathcal{S}(G)$ has a perfect order, then the order on G defined by this condition can be easily extended to a perfect order on G by introducing an arbitrary total order on every η-class.

(4) Follows immediately from (3) since $\mathcal{S}(G)=\mathcal{S}(H)$.

(5) Follows by the same reasoning as (3). □

Remark 5.4. The property of a graph to have a perfect order may be expressed by each of the following statements, which are easily proved to be equivalent.

(1) There is an order on G such that the set of cliques and the set of chains coincide.

(2) There is an order on G such that the set of maximal cliques and the set of maximal chains coincide.

(3) There is an order on G such that every clique is a chain and every anticlique is an antichain.

(4) There is an order on G such that the set of maximal anticliques and the set of maximal antichains coincide.

This property is also known under different names. Graphs with a perfect order are called *transitively orientable* [5, 7, 8] or *comparability graphs* [1, 7].

The next proposition has been obtained recently by Diekert [5].

Proposition 5.5. *For every graph $G = (A, \theta)$ the following conditions are equivalent:*

(1) *$\mu_M(G)$ can be lifted to $\mathbb{Z}\langle\langle A^*\rangle\rangle$,*

(2) *there is a perfect order on G.*

Remark 5.6. (1) If $<$ is a perfect order on $G = (A, \theta)$, then the set $\{(ba, ab) \mid a < b\}$ is a complete finite semi-Thue-system for $M(A, \theta)$ producing for every $w \in A^*$ the normal form $L(w)$. Conversely, let Ω be a complete finite semi-Thue-system for $M(A, \theta)$. Define $a < b$ if $(ba, ab) \in \Omega$. Since Ω has no loops, this defines a partial order on G. To see that this order is perfect, let $a, b, c \in A$ such that $a < b < c$ and consider $cba \in A^*$. From cba, Ω allows to derive bca and cab, hence we obtain from the local confluence of Ω that (ca, ac) must belong to Ω. This implies $a < c$ and thus $<$ is perfect. Perfect orderings on G are thus equivalent to complete semi-Thue-systems for $M(A, \theta)$. This fact has been noticed recently by Otto [8].

(2) There are graphs without perfect order: Every cycle of odd length greater than three is without perfect order.

Our investigations have shown that the essential algebraic properties of a free partially commutative monoid M are determined by the properties of the indecomposable graphs which are skeletal, whose complementary graph is skeletal, and which occur as parts of the graph describing M.

We conclude this section with a proposition simplifying the search for perfect orders on graphs. For a given graph $G = (A, \theta)$ we say that a system $L_1, L_2, \ldots, L_k$ of anticliques of G is a *vertex-cover* (of G or of A) by anticliques, if the L_i are anticliques of G and their union is A. Analogously we define vertex-covers by cliques. It is called minimal, if the number k is minimal. A closer inspection of the definition of the order on G induced by some cover of G by anticliques shows that

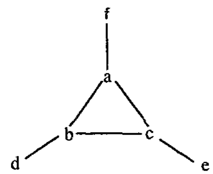


Fig. 8.

it is sufficient to have a minimal vertex-cover of G by anticliques to define the same order on G.

The graph F shown in Fig. 8 is an example showing that minimal covers by cliques and minimal vertex-covers by cliques have different cardinality in general. The system $\{a,f\}$, $\{b,d\}$, $\{c,e\}$ is the minimal vertex-cover of F, whereas $\{a,b,c\}$, $\{a,f\}$, $\{b,d\}$, $\{c,e\}$ is the only minimal cover of F by cliques.

Proposition 5.7. *Every perfect order on a graph G is induced by some canonical form of $\bar{G}$; in fact it is induced by some minimal vertex-cover of G by anticliques.*

Proof. For a given perfect order $<$ on $G=(A,\theta)$ we define the sequence $A_1, A_2, \ldots, A_k$ of subsets of A in the following way:
- $A_1 = \min A$,
- if $A-\bigcup_{1\leq j\leq i} A_j \neq \emptyset$, then $A_{i+1} = \min A-\bigcup_{1\leq j\leq i} A_j$.

It is clear that $A_1, A_2, \ldots, A_k$ is a partition of A and every A_i is an antichain. Evidently, the number k is the length of a maximal chain in the given order and hence $A_1, A_2, \ldots, A_k$ is a minimal cover of A by antichains. We complete this partition to a cover of A by antichains in the following way:

$$\text{for all } i=1,\ldots,k, \quad L_i = \max \bigcup_{1\leq j\leq i} A_j.$$

Then we have for all $i=1,\ldots,k$, $\bigcup_{1\leq j\leq i} L_j = \bigcup_{1\leq j\leq i} A_j$ and $A_i \subseteq L_i$, hence $L_1, L_2, \ldots, L_k$ form a cover of A by nonempty antichains and since every point in a maximal chain must belong to exactly one of the L_i, the system $L_1, L_2, \ldots, L_k$ is a minimal cover as well. Finally, every set L_i is a maximal antichain of A. In fact, suppose we could find some $a\in A-L_i$ such that $\{a\}\cup L_i$ is an antichain and i is minimal for this property. Then $i\neq 1$ since $L_1 = A_1$ is a maximal antichain. Furthermore, $a\notin \bigcup_{1\leq j\leq i} L_j = \bigcup_{1\leq j\leq i} A_j$, and $\{a\}\cup A_i$ is an antichain. This implies that $a\in \min A-\bigcup_{1\leq j\leq i-1} A_j = A_i$, contradicting the assumption $a\notin L_i$.

Since the order is perfect, the system $L_1, L_2, \ldots, L_k$ is a minimal vertex-cover of G by maximal anticliques, and the order induced by this vertex-cover is the original order. All the remaining maximal anticliques of G can now be appended to this set system without changing the induced order. $\square$

The previous proof has shown the following.

Corollary 5.8. *If a graph has a perfect order, then minimal vertex-covers and minimal (edge-) covers have the same cardinality.*

The graph $\bar{F}$ for the graph F defined above is an example for a graph having minimal vertex-cover and minimal cover with equal cardinality without having a perfect order. This shows that this condition does not characterize the class of graphs with perfect order.

Proposition 5.7 shows that looking for a perfect order on a graph G can be done by finding a minimal cover of G by anticliques with the property that in the induced order on G every maximal anticlique is an antichain. Hence we may take the vertex-cover of G by all maximal anticliques, i.e. the canonical form of $\bar{G}$, and test whether a permutation of this set system induces a perfect order. It is not clear whether it is sufficient to start with a minimal cover.

As we noticed earlier, the LR normal form with respect to some representation of $\bar{G}$ coincides with the alphabetic normal form with respect to the order induced by this representation. From Proposition 5.7 we also conclude that for every perfect order on G we have $T_{\mathrm{LR}} = T_L$, hence we can compute the alphabetic normal form without making explicit use of the order.

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